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# Path-integral quantization of Kaluza-Klein monopole systems

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A particle bound in the Kaluza-Klein monopole field (the static Taub-Newman-Unti-Tamburino space) is quantized by path integration. First, the system is regularized by the Kustaanheimo-Stiefel procedure. Then, path integration is performed in the Euler variables to separate the monopole harmonics. Dirac's charge quantization condition is deduced naturally by dimensional reduction. The radial path integral leads to the radial Green's function expressed in closed form, from which the discrete energy spectrum for $g<0$ ($g=4m$ is the monopole parameter) and the corresponding wave functions are obtained. A possibility of creating bound states for $g>0$ is also discussed by introducing an external Coulomb-like potential.

## I. INTRODUCTION

As is well known, the five-dimensional theory of Kaluza and Klein[1] is a unification scheme of gravity and electromagnetism. In the standard formulation of the theory, one of the spatial dimensions is assumed to be curled into a circle, and the metric of the five-dimensional space is taken to be independent of the fifth variable $x^5$. The line element of the space is given by

$$dw^2=V(x)[dx^5+A_\mu(x)dx^\mu]^2+g_{\mu\nu}(x)dx^\mu dx^\nu \ . \qquad (1.1)$$

Here both the scalar field $V(x)$ and the tensor field $g_{\mu\nu}(x)$ are understood as representing the effects of gravity, and the vector field $A_\mu(x)$ is identified with the electromagnetic potential. When the scalar field $V(x)$ is constant, we have a four-dimensional framework for the theory of gravitation with the $U(1)$ gauge theory of electrodynamics. In one of the earliest attempts to incorporate Dirac's monopole into the Kaluza-Klein scheme, Hoffmann[2] extended the space into the one in six dimensions and introduced another vector potential to describe the singular monopole potential together with the vector field of Kaluza and Klein. There have been a number of works on higher-dimensional Kaluza-Klein schemes which accommodate monopoles.[3] What is called the Kaluza-Klein monopole here refers to the simplest soliton solution of the classical field equation, $R_{AB}=0$, found by Gross and Perry,[4] and Sorkin;[5]

$$V(x)=(1+4m/r)^{-1}, \quad A_r=0,$$

$$A_\theta=0, \quad A_\phi=4m(\pm1-\cos\theta) \ . \qquad (1.2)$$

In the above, $R_{AB}$ is the Ricci tensor of the five-dimensional Kaluza-Klein space, $4m$ is the only parameter that characterizes the Kaluza-Klein monopole solution, and $(r,\theta,\phi)$ are polar coordinates in three-space. It has been recognized[4,5] that the subspace of (1.1) with $dx^0=0$ coincides with the static Taub-NUT (-Newman-Unti-Tamburino) instanton solution:[6,7]

$$dw^2=(1+4m/r)(dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2)$$
$$+(1+4m/r)^{-1}[dx^5+4m(\pm1-\cos\theta)d\phi]^2 \ . \qquad (1.3)$$

It has also been noted[8] that the line element (1.3) approximately describes the background for the relative motion of two Bogomol'nyi-Prasad-Sommerfield monopoles.[9] Quantization of a particle in such a space has been studied rather extensively; the correct energy spectrum has already been obtained not only via Schrödinger's equation, but also by the supersymmetric WKB calculation[10] and the geometric quantization method.[11]

In this paper we present the path-integral solution for a test-particle bound in the static Kaluza-Klein monopole geometry as given by (1.3). Because of the nontrivial structure of the background space, Feynman's path integral for the propagator cannot easily be evaluated. However, thanks to the recent development of various path-integration techniques, we have been able to solve a number of nontrivial examples including the Dirac monopole problem[12,13] and the Kepler problem in uniformly curved spaces by path integration.[14] Here we also take advantage of such new techniques. In Sec. II, employing the Kustaanheimo-Stiefel coordinates[15] and a new "time" parameter, we formulate the problem in terms of a path integral. Then, in Sec. III, we perform path integration explicitly by separating the monopole harmonics. Section IV deals with quantization of the dual charge $q=eg$. In Sec. V the radial Green's function is derived in closed form, from which we obtain the discrete energy spectrum as well as the wave functions for the Kaluza-Klein (KK) monopole system with a negative monopole parameter ($g=4m<0$). Section VI introduces an external path-integrable Coulomb-like potential which generates a discrete energy spectrum for the case of a positive parameter ($g>0$). Some remarks are made in Sec. VII regarding the new features of the path-integral treatment of the KK monopole.

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## II. PATH-INTEGRAL FORMULATION

The Lagrangian for the system of interest is

$$L=\frac{M}{2}\left[\frac{dw}{dt}\right]^2, \tag{2.1}$$

which is indeed for a test particle of mass $M$ moving in the static Taub-NUT space (1.3). The parameter $t$ employed in (2.1) differs from $x^0$ in (1.1), but refers to the time with which the particle's motion evolves in the curved background (1.3). From the Lagrange equations we can readily obtain the conserved quantity

$$e=M(1+rm/r)^{-1}[\dot{x}^5+4m(\pm 1-\cos\theta)\dot{\phi}], \tag{2.2}$$

which is usually interpreted as the electric charge. It is easy to see that the following Lagrangian is equivalent to the one associated with the Taub-NUT line element (2.1):

$$L=\frac{M}{2V}\dot{\mathbf{r}}^2+e\dot{x}^5+q(\pm 1-\cos\theta)\dot{\phi}-\frac{e^2}{2MV}, \tag{2.3}$$

where $\dot{\mathbf{r}}^2=\dot{r}^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2$ and $q=4me=eg$.

For path-integral quantization, however, the Lagrangian in either form (2.1) or (2.3) is too complicated. Since there is no reason to believe that quantization should be performed with the particular set of polar variables and time parameter chosen in (2.1), we may select those which are most convenient for path-integral quantization. In fact, it is known that special care has to be exercised in quantization in polar coordinates.[16,17] The set of coordinates we consider appropriate is the one of Kustaanheimo and Stiefel; that is, the Cartesian set $(u^1,u^2,u^3,u^4)$ related to Euler's angles $(\theta,\phi,\psi)$ by

$$\begin{aligned} u^1/u&=\cos(\theta/2)\cos[(\phi+\psi)/2],\\ u^2/u&=\cos(\theta/2)\sin[(\phi+\psi)/2],\\ u^3/u&=\sin(\theta/2)\cos[(\phi-\psi)/2],\\ u^4/u&=\sin(\theta/2)\sin[(\phi-\psi)/2], \end{aligned} \tag{2.4}$$

where $u=|\mathbf{u}|$. They satisfy the identities

$$r=u^2\equiv\mathbf{u}^2, \tag{2.5}$$

and

$$\dot{\mathbf{r}}^2+r^2(\dot{\psi}+\cos\theta\dot{\phi})^2=4u^2\dot{\mathbf{u}}^2. \tag{2.6}$$

In order to change the original variables into the Kustaanheimo-Stiefel (KS) coordinates, we further modify the Lagrangian (2.3) by introducing the additional variable $\psi$ as

$$L=\frac{M}{2V}[\dot{\mathbf{r}}^2+r^2(\dot{\psi}+\cos\theta\dot{\phi})^2]+e\dot{x}^5+q(\pm 1-\cos\theta)\dot{\phi}-\frac{e^2}{2MV}. \tag{2.7}$$

The added variable is cyclic, so that its conjugate momentum $\partial L/\partial\dot{\psi}$ is a constant of motion. Without loss of generality, we can demand the condition that the constant of motion associated with the $\psi$ variable is zero on a classical orbit. This demand is in fact the annihilation condition imposed by Kustaanheimo and Stiefel:[15]

$$2(-u^2du^1+u^1du^2+u^4du^3-u^3du^4)=0.$$

With the help of (2.5) and (2.6), the Lagrangian (2.7), which is equivalent to (2.1) and (2.3) under the condition imposed above, can be expressed in terms of the KS coordinates:

$$L=\frac{M}{2V}4u^2\dot{\mathbf{u}}^2+qh^{\pm}(\mathbf{u},\dot{\mathbf{u}})-\frac{e^2}{2MV}+e\dot{x}^5, \tag{2.8}$$

where $h^{\pm}(\mathbf{u},\dot{\mathbf{u}})=(\pm 1-\cos\theta)\dot{\phi}$, or, more explicitly,

$$h^{+}(\mathbf{u},\dot{\mathbf{u}})=\frac{2}{u^2}\left[u_3\dot{u}_4-u_4\dot{u}_3+\frac{u_3^2+u_4^2}{u_1^2+u_2^2}(u_1\dot{u}_2-u_2\dot{u}_1)\right],$$

$$h^{-}(\mathbf{u},\dot{\mathbf{u}})=-\frac{2}{u^2}\left[u_1\dot{u}_2-u_2\dot{u}_1+\frac{u_1^2+u_2^2}{u_3^2+u_4^2}(u_3\dot{u}_4-u_4\dot{u}_3)\right].$$

The change of variables alone does not help much for simplifying the Lagrangian. We wish also to choose an appropriate time parameter. First, we note that Hamilton's characteristic action integral $W=\int[L(q,\dot{q})+E]dt$ can yield the same equations of motion that result from Hamilton's principal action integral $S=\int L(q,\dot{q})dt$. Then we write the former action integral as

$$W=\int\mathcal{L}(q,\mathring{q})d\eta \tag{2.9}$$

where $\mathcal{L}(q,\mathring{q})=[L(q,\dot{q})+E](dt/d\eta)$, $\mathring{q}=dq/d\eta$, and $\eta$ is a new "time" parameter. For the regularization of the classical Kepler problem, Kustaanheimo and Stiefel used a time parameter $d\eta=dt/r$ that is proportional to the eccentric anomaly. Modifying the KS parameter, we choose $\eta$ such that

$$dt=4u^2d\eta/V(u^2). \tag{2.10}$$

Along a classical path, (2.10) is integrable, so that it is always possible to rescale the time interval $\tau=t''-t'$ by

$$\tau=4u''u'\sigma[V(u'^2)V(u''^2)]^{-1/2}, \tag{2.11}$$

where $\sigma=\eta''-\eta'$. Substitution of (2.8) and (2.10) into (2.9) yields

$$W^{(5)}=\int_\sigma\left[\frac{M}{2}\mathring{\mathbf{u}}^2+qh^{\pm}(\mathbf{u},\mathring{\mathbf{u}})-\frac{4e^2}{2M}\left[u^2+2g+\frac{g^2}{u^2}\right]+4E(u^2+g)\right]d\eta+e(x^{5\prime\prime}-x^{5\prime}), \tag{2.12}$$

which is certainly a qualified expression for the action of the classical Kaluza-Klein monopole system. With this action we shall carry out path-integral quantization for the KK system.

The path integral for the action (2.12) represents what is called the promotor[18] in five dimensions:

$$P^{(5)}(\mathbf{u}'',x^{5''};\mathbf{u}',x^{5'};\sigma)=\exp[(ie/\hbar)(x^{5''}-x^{5'})]P^{(4)}(\mathbf{u}'',\mathbf{u}';\sigma)\ , \tag{2.13}$$

where

$$P^{(4)}(\mathbf{u}'',\mathbf{u}';\sigma)=\int \exp\left[\frac{i}{\hbar}\int_\sigma W\,d\eta\right]\mathcal{D}[\mathbf{u}(\eta)]\ , \tag{2.14}$$

with

$$W=W^{(5)}-e(x^{5''}-x^{5'})\ . \tag{2.15}$$

The Green's function can be evaluated by integrating the four-dimensional promotor (2.14) as

$$G(\mathbf{u}'',\mathbf{u}';E)=\frac{1}{i\hbar}\int P^{(4)}(\mathbf{u}'',\mathbf{u}';\sigma)J(\tau,\sigma)d\sigma\ , \tag{2.16}$$

whose poles and residues provide us the energy spectrum and the wave functions of the system. The Jacobian $J(\tau,\sigma)=d\tau/d\sigma$ can be easily evaluated from (2.11).

## III. PERFORMING PATH INTEGRATION

Now we perform path integration for the promotor (2.14) explicitly. In the discretized version, the path integral (2.14) is given by

$$P^{(4)}(\mathbf{u}'',\mathbf{u}';\sigma)=\lim_{N\to\infty}\int\prod_{j=1}^{N}\left[\frac{M}{2\pi i\hbar\sigma_j}\right]^2\prod_{j=1}^{N}e^{(i/\hbar)W_j}\prod_{j=1}^{N-1}d\mathbf{u}_j\ , \tag{3.1}$$

with

$$W_j=\frac{M}{2\sigma_j}(\Delta\mathbf{u}_j)^2+qh_j^{\pm}\sigma_j-\left[\frac{2e^2}{M}-4E\right]\hat{u}_j^2\sigma_j-\frac{2q^2}{M\hat{u}_j^2}\sigma_j+4g\left[E-\frac{e^2}{M}\right]\sigma_j\ . \tag{3.2}$$

In the above we have used the notation $\Delta\mathbf{u}_j=\mathbf{u}_j-\mathbf{u}_{j-1}$, $\hat{u}_j^2=u_ju_{j-1}$, and $\sigma=\sum_{j=1}^{N}\sigma_j$. For explicit path integration, however, it is more convenient to use $u$ and the Euler angles of (2.4), in terms of which the path integral (3.1) can be expressed as

$$P^{(4)}(\mathbf{u}'',\mathbf{u}';\sigma)=e^{\pm iq(\phi''-\phi')}\exp\left[\frac{4ig}{\hbar}\left[E-\frac{e^2}{M}\right]\sigma\right]\lim_{N\to\infty}\int\prod_{j=1}^{N}\left[\frac{M}{2\pi i\hbar\sigma_j}\right]^2\prod_{j=1}^{N}e^{(i/\hbar)S_j}\prod_{j=1}^{N-1}\frac{1}{8}u_j^3\sin\theta_j du_j d\theta_j d\phi_j d\psi_j\ , \tag{3.3}$$

with the short-time action

$$S_j=\frac{M}{2\sigma_j}(\Delta u_j)^2+\frac{M\hat{u}_j^2}{\sigma_j}\left[1-\cos\frac{\Theta_j}{2}\right]-q\cos\bar{\theta}_j\Delta\phi_j-\left[\frac{2e^2}{M}-4E\right]\hat{u}_j^2\sigma_j-\frac{4q^2}{2M\hat{u}_j^2}\sigma_j\ , \tag{3.4}$$

where

$$\cos\frac{\Theta_j}{2}=\cos\frac{\theta_j}{2}\cos\frac{\theta_{j-1}}{2}\cos\frac{\Delta\phi_j+\Delta\psi_j}{2}+\sin\frac{\theta_j}{2}\sin\frac{\theta_{j-1}}{2}\cos\frac{\Delta\phi_j-\Delta\psi_j}{2}\ ,$$

$$\cos\bar{\theta}_j=\cos\frac{\theta_j}{2}\cos\frac{\theta_{j-1}}{2}-\sin\frac{\theta_j}{2}\sin\frac{\theta_{j-1}}{2}\ ; \tag{3.5}$$

and $\Delta u_j=u_j-u_{j-1}$, $\Delta\phi_j=\phi_j-\phi_{j-1}$. The angular terms in (3.4) may be combined together as

$$\begin{aligned}\frac{M\hat{u}_j^2}{\sigma_j}\cos\left[\frac{\Theta_j}{2}\right]+q\cos\bar{\theta}_j\Delta\phi_j=&\frac{M\hat{u}_j^2}{\sigma_j}\cos\frac{\theta_j}{2}\cos\frac{\theta_{j-1}}{2}\left[\cos\left[\frac{\Delta\phi_j+\Delta\psi_j}{2}\right]+\frac{2q\sigma_j}{M\hat{u}_j^2}\left[\frac{\Delta\phi_j+\Delta\psi_j}{2}-\frac{\Delta\psi_j}{2}\right]\right]\\&+\frac{M\hat{u}_j^2}{\sigma_j}\sin\frac{\theta_j}{2}\sin\frac{\theta_{j-1}}{2}\left[\cos\left[\frac{\Delta\phi_j-\Delta\psi_j}{2}\right]-\frac{2q\sigma_j}{M\hat{u}_j^2}\left[\frac{\Delta\phi_j-\Delta\psi_j}{2}+\frac{\Delta\psi_j}{2}\right]\right]\ .\end{aligned} \tag{3.6}$$

The approximation trick $\cos\Delta\phi\pm\delta\Delta\phi=\cos(\Delta\phi\mp\delta)+\delta^2/2+O(\delta^3)$, as used in Ref. 19, enables us to write the right-hand side of (3.6) in the form

$$\frac{M\hat{u}_j^2}{\sigma_j}\cos\left[\frac{\tilde{\Theta}_j}{2}\right]+\frac{4q^2}{2M\hat{u}_j^2}\sigma_j\cos\left[\frac{\Delta\theta_j}{2}\right]-q\,\Delta\psi_j\cos\left[\frac{\Delta\theta_j}{2}\right],$$

where

$$\cos(\tilde{\Theta}_j/2)=\cos(\theta_j/2)\cos(\theta_{j-1}/2)\cos[(\Delta\phi_j+\Delta\tilde{\psi}_j)/2]+\sin(\theta_j/2)\sin(\theta_{j-1}/2)\cos[(\Delta\phi_j-\Delta\tilde{\psi}_j)/2]\ . \tag{3.7}$$

Here we have introduced the shifted angle $\Delta\tilde{\psi}_j=\Delta\psi_j-4q\sigma_j/M\hat{u}_j^2$. We further take account of the approximate relations $\sigma_j\cos(\Delta\theta_j/2)=\sigma_j+O(\sigma_j^2)$ and $\Delta\psi_j\cos(\Delta\theta_j/2)=\Delta\psi_j+O(\sigma_j^{3/2})$ to express the short-time action (3.4) in an equivalent form:

$$S_j=\frac{M}{s\sigma_j}(\Delta u_j)^2\frac{M\hat{u}_j^2}{\sigma_j}\left[1-\cos\frac{\tilde{\Theta}_j}{2}\right]-\left[\frac{2e^2}{M}-4E\right]\hat{u}_j^2\sigma_j-\frac{8q^2}{2M\hat{u}_j^2}\sigma_j+q\,\Delta\psi_j\ . \tag{3.8}$$

Next, we use the unitary irreducible representations of SU(2),

$$\mathcal{D}^l_{\mu\nu}(\phi,\theta,\tilde{\psi})=e^{-i\mu\phi}d^l_{\mu\nu}(\theta)e^{-i\nu\tilde{\psi}}\ ,$$

to make the following expansion of the angular contribution:[20]

$$\left[\frac{M\hat{u}_j^2}{2\pi i\hbar\sigma_j}\right]^{3/2}\exp\left\{\frac{iM\hat{u}_j^2}{\hbar\sigma_j}\left[1-\cos\left[\frac{\tilde{\Theta}_j}{2}\right]\right]\right\}$$
$$=\sum_{2l=0}^{\infty}\sum_{\mu\nu=-l}^{l}\frac{2l+1}{2\pi^2}\exp\left[-\frac{(2l+1)^2-8\nu q/\hbar-\frac{1}{4}}{2M\hat{u}_j^2}i\hbar\sigma_j\right]e^{-i\mu\Delta\phi_j}e^{-i\nu\Delta\psi_j}d^l_{\mu\nu}(\theta_j)d^{l*}_{\mu\nu}(\theta_{j-1})\ . \tag{3.9}$$

Carrying out the angular integrations of (3.3) with the aid of (3.9) and the orthogonality relations of the Wigner functions,[20] we find the promotor in the form

$$P^{(4)}(\mathbf{u}'',\mathbf{u}';\sigma)=\sum_{2l=0}^{\infty}\sum_{\mu,\nu=-l}^{l}\frac{2l+1}{16\pi^2}e^{-i(\mu\mp q/\hbar)(\phi''-\phi')}e^{-i(\nu-q/\hbar)(\psi''-\psi')}d^l_{\mu\nu}(\theta'')d^{l*}_{\mu\nu}(\theta')P^{(4)}_l(u'',u';\sigma)\ , \tag{3.10}$$

where the radial promotor

$$P^{(4)}_l(u'',u';\sigma)=8(u''u')^{-1/2}\exp[(4ig/\hbar)(E-e^2/M)\sigma]\tilde{K}_\lambda(u'',u';\sigma) \tag{3.11}$$

is given as a radial path integral in four-space:

$$\tilde{K}_\lambda(u'',u';\sigma)=\frac{1}{u'u''}\lim_{N\to\infty}\int\prod_{j=1}^{N}\left[\frac{M}{2\pi i\hbar\sigma_j}\right]^{1/2}\exp\left[\frac{i}{\hbar}\tilde{S}_j\right]\prod_{j=1}^{N-1}du_j\ , \tag{3.12}$$

with

$$\tilde{S}_j=\frac{M}{2\sigma_j}(\Delta u_j)^2-\lambda(\lambda+1)\frac{\hbar^2\sigma_j}{2M\hat{u}_j^2}-\frac{M\omega^2}{2}\hat{u}_j^2\sigma_j\ . \tag{3.13}$$

In the above we have set $\lambda=[(2l+1)^2-(8q/\hbar)(\nu-q/\hbar)]^{1/2}-\frac{1}{2}$ and

$$\omega^2=4e^2/M^2-8E/M\ . \tag{3.14}$$

The last path integral (3.12) is identical in form with that for the radial propagator of the three-dimensional harmonic oscillator which has been evaluated exactly, the result being[17]

$$\tilde{K}_\lambda(u'',u';\sigma)=(u''u')^{-1/2}\frac{M\omega}{i\hbar}\csc(\omega\sigma)\exp\left[\frac{iM\omega}{2\hbar}(u''^2+u'^2)\cot(\omega\sigma)\right]I_{\lambda+1/2}\left[\frac{M\omega u''u'}{i\hbar}\csc(\omega\sigma)\right]\ . \tag{3.15}$$

In this fashion we have completed the path integration for the system represented by the action integral (2.12). There are still a few more steps we have to take before reaching the Green's function in three-space, from which we can read off physically meaningful information.

## IV. CHARGE QUANTIZATION

Before evaluating the Green's function, we have to reduce the promotor (3.10) to the one in three-space, eliminating the extra degree of freedom by integration:

$$P^{(3)}(\mathbf{r}'',\mathbf{r}';\sigma)=u''u'\int_0^{4\pi}P^{(4)}(\mathbf{u}'',\mathbf{u}';\sigma)d\psi''\ . \tag{4.1}$$

The integration over the extra variable $\psi''$ in (4.1) is as simple as

$$\int_0^{4\pi} e^{-i(\nu - q/\hbar)\psi''} d\psi'' = 4\pi\delta_{\nu,q/\hbar} , \tag{4.2}$$

which results, as $\nu$ is a half-integer, in charge quantization:

$$q = eq = s\hbar, \quad 2s = 0, \pm 1, \pm 2, \ldots \ . \tag{4.3}$$

This coincides with the well-known charge quantization condition of Dirac.[21]

Energy quantization and charge quantization are in nature two independent processes. For the KK monopole system, the former is quantization of the energy of the content (a test particle) in a fixed Taub-NUT geometry, whereas the latter is in a way quantization of the container (quantization of a parameter characterizing the geometry). In quantizing the KK system, therefore, the dual charge $q$ is fixed to a constant value since each geometry is specified by a single value of $q$. For charge quantization the cylindrical condition is usually employed, which demands that the $x^5$ coordinate is periodic. The present path-integral approach not only provides the energy spectrum and the energy eigenfunctions, but also has a built-in scheme that leads us very naturally to Dirac's quantization condition (4.1). This charge quantization by the dimensional reduction is indeed a topological quantization.

## V. ENERGY SPECTRUM AND WAVE FUNCTIONS

The promotor (3.10) with (3.11) and (3.15) may be put into the form

$$P^{(3)}(\mathbf{r}'', \mathbf{r}'; \sigma) = \sum_{l=|s|}^{\infty} \sum_{\mu=-l}^{l} P_l(r'', r'; \sigma) Y_{s,l,\mu}^{(\pm)}(\theta'', \phi'') Y_{s,l,\mu}^{(\pm)*}(\theta', \phi') , \tag{5.1}$$

where

$$Y_{s,l,\mu}^{(\pm)}(\theta, \phi) = \left[ \frac{2l+1}{4\pi} \right]^{1/2} d_{\mu,s}^{l}(\theta) e^{-i(\mu \mp s)\phi} \tag{5.2}$$

are, up to an arbitrary phase factor, identical with the monopole harmonics of Wu and Yang.[22] Note that the first sum in (5.1) covers either all integers or half-integers depending on whether $s$ takes integral values or half-integral values. The radial promotor is given by (3.11) and (3.15) with $r'' = u''^2$ and $r' = u'^2$. Note also that $\lambda = 2l + \frac{1}{2}$.

The Green's function is obtained by integration over the time interval $\sigma$:

$$G_l(r'', r'; E) = \frac{N}{i\hbar} \int_0^\infty P_l(r'', r'; \sigma) J(\tau, \sigma) (r'' r')^{-1/2} d\sigma , \tag{5.3}$$

where $N$ is a normalization constant. The path-integral treatment usually provides a correct normalization because of the initial condition $\lim_{N \to \infty} K(\mathbf{r}'', \mathbf{r}'; \tau) = \delta(\mathbf{r}'' - \mathbf{r}')$. However, the present regularization procedure, being applied at the classical level, does not guarantee the normalization of the Green's function $(E - H)G = 1$. Therefore, we insert $N$ in (5.3). Setting $\omega\sigma = -iz$, $x = 2ikr''$, $y = 2ikr'$, and

$$M\omega = 2i\hbar k, \quad p = \frac{igM}{\hbar^2 k} \left[ E - \frac{e^2}{M} \right] = \frac{2g}{\hbar\omega} \left[ \frac{e^2}{M} - E \right] , \tag{5.4}$$

and utilizing the integral formula[23]

$$\int_0^\infty e^{-2xp} \exp[-\tfrac{1}{2}(x+y)\coth z] I_{2\nu}[(xy)^{1/2} \operatorname{csch} z] \operatorname{csch} z \, dz = [\Gamma(p + \nu + \tfrac{1}{2})/(xy)^{1/2}\Gamma(2\nu+1)] M_{-p,\nu}(x) W_{-p,\nu}(y) , \tag{5.5}$$

we obtain the radial Green's function in closed form:

$$G_l(r'', r'; E) = -\frac{32MN}{\hbar^2} \frac{[(1 + 4m/r'')(1 + 4m/r')]^{1/2}}{2ikr''r'} \frac{\Gamma(p + l + 1)}{(2l+1)!} M_{-p,l+1/2}(2ikr') W_{-p,l+1/2}(2ikr'') , \tag{5.6}$$

where $M_{\alpha,\beta}(z)$ and $W_{\alpha,\beta}(z)$ are the Whittaker functions.

The poles of (5.6) occur only when in $\Gamma(p + l + 1)$,

$$p + l + 1 = -n_r, \quad n_r = 0, 1, 2, \ldots , \tag{5.7}$$

which give rise to the discrete energy spectrum for which we are looking:

$$\begin{aligned} E_n &= \frac{\hbar^2}{g^2 M} [\pm n (n^2 - s^2)^{1/2} - (n^2 - s^2)] \\ &= \frac{e^2}{M} \left[ \pm \frac{n}{|s|} \left[ \frac{n^2}{s^2} - 1 \right]^{1/2} - \left[ \frac{n^2}{s^2} - 1 \right] \right] , \end{aligned} \tag{5.8}$$

where $s=q/\hbar$ and $n=n_r+l+1>s$. The radial Green's function (5.6) also possesses a branch cut in the $E$ plane along a real line $(e^2/2M)\le E$, which gives rise to a continuous spectrum

$$E=\frac{e^2}{2M}+\frac{\hbar^2k^2}{2M}\ , \quad k\in\mathbb{R}\ . \tag{5.9}$$

The radial wave functions corresponding to the discrete spectrum (5.8) can be obtained from the residues of the Green's function at the poles. Using the relations

$$\mathrm{Res}\Gamma(\epsilon)|_{\epsilon=-n_r}=(-1)^{n_r}/n_r!\ ,$$

$$W_{n_r+l+1,l+1/2}(z)=(-1)^{n_r}n_r!z^{l+1}e^{-z/2}L_{n_r}^{(2l+1)}(z)\ ,$$

$$M_{n_r+l+1,l+1/2}(z)=\frac{n_r!\Gamma(2l+2)}{\Gamma(n_r+2l+2)}z^{l+1}e^{-z/2}L_{n_r}^{(2l+1)}(z)\ ,$$

and setting $N=(gME/16e\hbar p)$, we find

$$\mathrm{Res}G_l(r'',r';E)|_{E=E_n}=R_{nl}(r'')R^*_{nl}(r')\ , \tag{5.10}$$

where

$$\begin{aligned}R_{nl}(r)&=\left[\frac{4(n-l-1)!}{a_n^3n(n+l)!}\right]^{1/2}\left[\frac{2r}{a_n}\right]^l\left[1+\frac{4m}{r}\right]^{1/2}e^{-r/a_n}L_{n-l-1}^{(2l+1)}\left[\frac{2r}{a_n}\right]\\&=\left[\frac{4}{a_n^3n}\frac{(n+l)!}{(n-l-1)!}\right]^{1/2}\frac{(2r/a_n)^l}{(2l+1)!}\left[1+\frac{4m}{r}\right]^{1/2}e^{-r/a_n}{}_1F_1\left[-n+l+1,2l+2;\frac{2r}{a_n}\right]\ ,\end{aligned} \tag{5.11}$$

and $a_n=\hbar(e^2-2E_nM)^{1/2}$. The radial wave functions $R_{nl}(r)$ of (5.11), which are normalized with the measure $(1+4m/r)^{-1}r^2dr$, vanish at $r=-4m$ unless $s=0$ $(4m=0)$. It is interesting to observe that the radial functions (5.11) are identical, apart from the factor $(1+4m/r)^{1/2}$, with the radial wave functions of the hydrogen atom in flat space if $(a_n/n)$ is replaced by the Bohr radius. The complete wave functions for the bound states are given by

$$\psi^{\pm}_{n,l,\mu}(\mathbf{r},x^5)=(16m\pi)^{-1/2}R_{nl}(r)Y^{\pm}_{s,l,\mu}(\theta,\phi)\exp(isx^5/4m)\ , \tag{5.12}$$

which are normalized with measure $(1+4m/r)^{-1}r^2\sin\theta\, dr\, d\theta\, d\phi\, dx^5$, and $x^5\in[0,16m\pi)$.

At this point we wish to make a remark concerning the discrete energy spectrum (5.8). From (5.4) and (5.7) it is clear that since $p<0$ the bound states $(E<e^2/2M)$ are possible only when $g=4m<0$. Under this condition the energy spectrum (5.8) has two branches: $(e^2/2M)>E_n>0$ and $E_n<0$. Strangely, two energy eigenvalues correspond to a single value of the principal quantum number $n$. Furthermore, while the positive spectrum has an upper limit, the negative spectrum has no lower limit.

To understand the strange structure of the energy spectrum, we have to remind ourselves of the fact that the singular radius $r=-4m$ imposes a strong restriction on the classical motion of a test particle in the Taub-NUT space. A particle of positive energy is allowed only in the exterior region $(r>-4m)$, whereas a particle in the interior $(r<-4m)$ is limited to possess a negative energy. An exterior particle, which is under the influence of an attractive Coulomb-like effective potential, is bound if $0\le E<e^2/2M$ and unbound if $E\ge e^2/2M$. The exterior particle, possessing a positive energy, cannot penetrate into the interior region where only negative-energy particles are permitted to exist. The effective potential for an interior particle is always repulsive, so that no bound state is possible. Nevertheless, the particle of negative energy cannot come out without acquiring an infinite amount of energy into the exterior region. Consequently, any test particle inside will be pushed onto the spherical wall $r=-4m$. The sphere $r=-4m$ is in fact the attractor which an interior particle will approach. The energy spectrum (5.8) reflects this classical situation. The positive branch of the energy spectrum corresponds to the classical particle bound in the Coulomb-like potential in the exterior region. This explains why the radial wave function (5.11) resembles those of the hydrogen atom. It fact, the positive branch can be reduced to the usual hydrogen-atom spectrum with an additive constant for large quantum numbers $n$ if $(es\hbar/2M)^{1/2}$ is identified with the Coulomb charge. In contrast with the classical situation, in quantum mechanics, the wave functions (5.11) for $E_n>0$ penetrate into the interior region to reach the origin $r=0$ where they vanish if $l>\frac{1}{2}$ and remain finite if $l=\frac{1}{2}$. Now the negative branch of the spectrum can be interpreted as belonging to the classical particle confined in the interior region. The wave functions of the interior particle can also extend their tails to infinity beyond the singular sphere at $r=-4m$. It must be noted, however, that the wave functions (5.11) belonging to the positive-energy spectrum have the constants $a_n$ specified by the positive values of $E_n$. Those belonging to the negative-energy spectrum have $a_n$ determined by the negative values of $E_n$, which are much smaller than those for $E_n > 0$. Because of the smallness of the values of $a_n$, the wave functions for $E_n < 0$ diminish quickly in the exterior region.

## VI. INTEGRABLE MODEL

The Kaluza-Klein system we have considered so far has a discrete energy spectrum only for $g = 4m < 0$. Let us now pursue a possibility of creating bound states for $g > 0$. Introducing an external Coulomb-like potential to the original Lagrangian (2.1), we consider a simple path-integrable model that gives for a positive monopole charge a discrete energy spectrum which reduces to the hydrogenlike spectrum with an additive constant when $g$ is small.

Namely, we modify the Lagrangian (2.7) for the pure KK system as

$$\begin{aligned} L^c &= L + \frac{gE_0 V}{r} \\ &= \frac{M}{2V}\left[\dot{\mathbf{r}}^2 + r^2(\dot{\psi} + \cos\theta\dot{\phi})^2\right] \\ &\quad + e\dot{x}^5 + q(\pm 1 - \cos\theta)\dot{\phi} - \frac{e^2}{2MV} + \frac{gE_0 V}{r}\ , \end{aligned} \tag{6.1}$$

where $E_0 \geq e^2/2M$. The short-time action corresponding to (3.2), expressed in KS coordinates, reads

$$\begin{aligned} W_j^c &= W_j + 4gE_0\sigma_j \\ &= \frac{M}{2\sigma_j}(\Delta \mathbf{u}_j)^2 + qh_j^{\pm}\sigma_j \\ &\quad - \left[\frac{2e^2}{M} - 4E\right]\hat{u}_j^2\sigma_j - \frac{2q^2}{M\hat{u}_j^2}\sigma_j \\ &\quad + 4g\left[E + E_0 - \frac{e^2}{M}\right]\sigma_j\ . \end{aligned} \tag{6.2}$$

Apparently, the extra potential gives rise to an additional phase factor $\exp(4igE_0\sigma/\hbar)$ in the promotor (3.3). Hence the calculation in Secs. III–V remains the same. The additional phase only affects on the parameter $p$ in (5.4) as

$$p^c = p - \frac{2g}{\hbar\omega}E_0 = \frac{2g}{\hbar\omega}\left[\frac{e^2}{M} - E - E_0\right]. \tag{6.3}$$

The poles of the Green's function (5.6) with such a modification occur now at $p^c = -n$ for $g > 0$. The resultant discrete energy spectrum is

$$E_n^c = \frac{e^2}{M}\left[\frac{n}{|s|}\left[\frac{n^2}{s^2} - 1 + 2\epsilon_0\right]^{1/2} - \left[\frac{n^2}{s^2} - 1 + \epsilon_0\right]\right], \tag{6.4}$$

with $\epsilon_0 = ME_0/e^2$, and has the following range: $e^2/M - E_0 \leq E_n^c < e^2/2M$. Note that the lower bound eliminates the second branch of the spectrum appearing in (5.8).

## VII. CONCLUDING REMARKS

In the above we have discussed the path-integral treatment of the modified as well as pure KK systems. As for the pure KK system with $4m < 0$, the bound-state energies have already been obtained by a variety of methods.[8,10,11] However, no explicit expressions for the energy eigenfunctions have been given, though expected in the form of confluent hypergeometric functions. The present path-integral method certainly provides yet another way of handling the KK monopole. In most of the other approaches, the Hamiltonian, constructed on the Taub-NUT space, is the central object. In contrast, the path-integral deals with the Lagrangian directly formed from the line element of the background space. It is as clear as in the other treatments that the bound states occur only when the magnetic charge $g$ is negative. The resultant discrete spectrum for the bound states is in agreement with the one already available. The new features emerging from the present path-integral treatment are the natural quantization condition on the dual charge $q = eg$, the two branches of the energy spectrum and the explicit expression for the corresponding wave functions, which vanish at $r = -4m$. The monopole harmonics for the angular parts are separated from the radial part, and the radial part is path-integrated explicitly and given in closed form in terms of the Whittaker functions.

Application of the Kustaanheimo-Stiefel regularization in a path integral is not all new. The KS procedure has been used previously in solving the hydrogen-atom problem by path integration.[24,25] Since it has been directly implemented in the Coulomb path integral, there are still unresolved ambiguities in calculation. For instance, the Jacobian of the KS coordinate mapping from $R^4$ to $R^3$ does not exist, so that the measure of path integration cannot be determined uniquely. Furthermore, the position-dependent time transformation may be integrated along the classical path, but it is difficult to understand how one can arrive at a unique value for the total time interval by integration over all paths. To circumvent such ambiguous problems, we have implemented the KS procedure at the classical level and applied the path-integral quantization to the KK system described in the KS rectangular coordinates. From the technical aspect, this is also a feature of the present path-integral treatment.

To seek a possibility of bound states for the case of $g > 0$, we have also considered the influence of an external Coulomb-like potential on the KK system. In fact, we have found it possible to create a discrete energy spectrum for the positive monopole charge. Although the pure KK monopole is a solution of the field equation $R_{AB} = 0$ in five-dimensional Kaluza-Klein space, the modified KK system is not. Whether the modified system can be found as a solution for the field equation with an appropriate source term is left for future study.

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